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On the composition of q-skew derivations in Banach algebras

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ABSTRACT

Let $\mathcal{A}$ be a Banach algebra, let σ be an automorphism of $\mathcal{A}$, and let d, δ be q-skew σ-derivations of $\mathcal{A}$. We show that if $d\delta(a)$ is quasi-nilpotent for any $a \in \mathcal{A}$, then $d\delta(a)^3$ lies in the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$. This result simultaneously generalizes the previous results obtained by Brešar and Šemrl [5], Chebotar et al. [7] and Abdelali [1].

1. Introduction

Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$. For $a, b \in \mathcal{A}$, let $[a, b] = ab - ba$ be the commutator of a and b. The classical Kleinecke–Shirokov theorem [16,21] states that if a and b are elements in $\mathcal{A}$ such that $[b, [b, a]] = 0$, then $[b, a]$ is *quasi-nilpotent*, namely, it has 0 spectral radius. Put another way, if d is the inner derivation defined by $d(x) = [b, x]$ for $x \in \mathcal{A}$ such that $d^2(a) = 0$ for some element $a \in \mathcal{A}$, then $d(a)$ is quasi-nilpotent. This result was extended later to continuous derivations by Mathieu and Murphy [18] and to arbitrary derivations by Thomas [23].

On the other hand, Pták [19] proved that if d is an inner derivation of $\mathcal{A}$ such that $d^2(a)$ is quasi-nilpotent for every $a \in \mathcal{A}$, then $d^2(a)^2$ lies in the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$. Then Turovskii and Shul'man [24] extended this result to arbitrary derivations. In [5] Brešar and Šemrl characterized commuting pairs of continuous derivations d, δ of $\mathcal{A}$ such that $d\delta(a)$ is quasi-nilpotent for any $a \in \mathcal{A}$. In particular,

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it turns out that $d\delta(a)^3$ must lie in the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$. Later, Chebotar et al. [7] generalized this result by removing the commutativity assumption $d\delta = \delta d$ and proved: If d, δ are continuous derivations of $\mathcal{A}$ such that $d\delta(a)$ is quasi-nilpotent for any $a \in \mathcal{A}$, then $d\delta(a)^3$ lies in the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$. A closely related result was recently proved by Boudi [3] who showed that if d, δ are continuous derivations of $\mathcal{A}$ such that $d\delta(a)$ has finite spectrum for any $a \in \mathcal{A}$, then $d\delta(a)^3$ lies in the socle of $\mathcal{A}$ modulo the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$.

Let σ be an automorphism of $\mathcal{A}$. By a σ-derivation δ of $\mathcal{A}$ we mean a linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{A}$. Clearly, the $1_{\mathcal{A}}$-derivations are just the ordinary derivations where $1_{\mathcal{A}}$ is the identity automorphism of $\mathcal{A}$. The σ-derivations are also called *skew* derivations. Among the skew derivations one important class is that of q-skew derivations as introduced in [11]. A σ-derivation δ of $\mathcal{A}$ is called *q-skew* if $\sigma\delta\sigma^{-1} = q\delta$, where $0 \neq q \in \mathbb{C}$. Clearly, the 1-skew σ-derivations are just the σ-derivations that commute with the basic automorphism σ. The q-skew derivations appear in q-Weyl algebras, enveloping algebras of solvable Lie superalgebras, and coordinate rings of quantum matrices. See [2,8,9,12–14,17] for some recent results concerning the q-skew derivations. In this paper, we shall generalize Brešar–Šemrl theorem to arbitrary q-skew derivations without assumptions on continuity or commutativity. A special case of continuous 1-skew derivations was recently proved by Abdelali [1] under some extra commutativity assumptions.

More precisely, our main result in this paper is as follows:

Theorem 1.1. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let d, δ be q-skew σ-derivations of $\mathcal{A}$. If $d\delta(a)$ is quasi-nilpotent for any $a \in \mathcal{A}$, then $d\delta(a)^3$ lies in the radical of $\mathcal{A}$ for all $a \in \mathcal{A}$.*

2. Preliminaries

Recall that the radical of an algebra $\mathcal{A}$ is equal to the intersection of the kernels of all irreducible representations of $\mathcal{A}$. Thus, in order to prove Theorem 1.1, it suffices to show that $\pi d\delta(a)^3 = 0$ for all $a \in \mathcal{A}$ and for any irreducible representation π of $\mathcal{A}$.

Let π be an irreducible representation of the Banach algebra $\mathcal{A}$ on a vector space X. That is, π is an algebra homomorphism of $\mathcal{A}$ into $L(X)$, the algebra of all linear operators on X, such that $\{0\}$ and X are the only subspaces invariant with respect to $\pi(\mathcal{A})$. Let x_0 be any fixed nonzero element of X and define

$$\|x\| = \inf\{\|a\| \mid a \in \mathcal{A}, \pi(a)x_0 = x\}$$

for each $x \in X$. Then it can be shown that $\|\cdot\|$ is a complete norm on X and $\|\pi(a)x\| \leq \|a\|\|x\|$ for all $a \in \mathcal{A}$ and all $x \in X$. (See the proof of [20, Theorem 2.2.6].) Thus $\pi(\mathcal{A}) \subseteq B(X)$, the Banach algebra of all linear operators on X which are bounded with respect to the norm $\|\cdot\|$. Hence, $\|\pi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$ and so $\pi : \mathcal{A} \to B(X)$ is continuous.

From now on, π will always denote a continuous irreducible representation of the Banach algebra $\mathcal{A}$ on a Banach space X with the norm defined as above for some fixed $x_0 \in X$. (Note that any two such norms are mutually equivalent. See the proof of [20, Theorem 2.2.7]. So $B(X)$ is well defined by any one of such norms.) Following [4] we call an automorphism σ of $\mathcal{A}$ a *π-inner* automorphism if there exists an invertible $P \in L(X)$ such that $\pi\sigma(a) = P\pi(a)P^{-1}$ for all $a \in \mathcal{A}$. An automorphism not π-inner is called *π-outer*. Similarly a σ-derivation δ of $\mathcal{A}$ is called *π-inner* if there exists $Q \in L(X)$ such that $\pi\delta(a) = \pi\sigma(a)Q - Q\pi(a)$ for all $a \in A$. Otherwise, it is called *π-outer*. For a σ-derivation δ, if both σ and δ are π-inner, such operators P, P^{-1} and Q are automatically bounded as shown below. The following proposition is a generalization of [22, Corollary 3.6] without continuity assumptions.

Proposition 2.1. *Let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. Suppose that there exist $P, Q \in L(X)$ with P invertible in $L(X)$ such that $\pi\sigma(a) = P\pi(a)P^{-1}$ and $\pi\delta(a) = \pi\sigma(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. Then P, P^{-1} and Q are all bounded.*

Proof. Clearly, if P, P^{-1} and Q are all bounded, then both $\pi\sigma$ and $\pi\delta$ are continuous. First we verify the continuity of $\pi\sigma$ and $\pi\delta$.

Let $\mathcal{P}$ be the kernel of π; then $\mathcal{P}$ is a closed ideal of $\mathcal{A}$ so that $\mathcal{A}/\mathcal{P}$ is also a normed algebra. If $\mathcal{P}$ has infinite codimension, then $\pi\delta$ is continuous by [6, Lemma 4.1]. So assume that $\mathcal{P}$ has finite codimension. Since σ and δ are both π-inner, we have $\sigma(\mathcal{P}) \subseteq \mathcal{P}$ and $\delta(\mathcal{P}) \subseteq \mathcal{P}$. Thus the map $\overline{\pi\delta} : \mathcal{A}/\mathcal{P} \to B(X)$ defined by $\overline{\pi\delta}(\overline{a}) = \pi\delta(a)$, where $\overline{a} = a + P \in \mathcal{A}/\mathcal{P}$, is well defined. Now $\mathcal{A}/\mathcal{P}$ is finite-dimensional, and it is well known that any linear map from a finite-dimensional normed space into another normed space must be continuous. Thus, $\overline{\pi\delta}$ is continuous. Let (a_n) be a sequence in $\mathcal{A}$ with $\lim_{n\to\infty} a_n = 0$. Then $\lim_{n\to\infty} \overline{a_n} = \overline{0}$, where $\overline{a_n} = a_n + \mathcal{P}$, and so $\lim_{n\to\infty} \pi\delta(a_n) = \lim_{n\to\infty} \overline{\pi\delta}(\overline{a_n}) = 0$. Hence, $\pi\delta$ is continuous.

Note that $\sigma - 1_{\mathcal{A}}$ is a σ-derivation of $\mathcal{A}$ and is also π-inner since $\pi(\sigma - 1_{\mathcal{A}})(a) = \pi\sigma(a)I - I\pi(a)$ for all $a \in \mathcal{A}$, where I is the identity operator on X. Hence, by the preceding paragraph, $\pi(\sigma - 1_{\mathcal{A}})$ is continuous and then so is $\pi\sigma$.

Now we show that both P and P^{-1} are bounded. Fix a nonzero $x_0' \in X$ and set $x_0 = P^{-1}x_0'$. Define a complete norm on X by $\|x\| = \inf\{\|a\| \mid a \in \mathcal{A}, \pi(a)x_0 = x\}$ for $x \in X$. Then, for $x \in X$ and $a \in \mathcal{A}$ with $\pi(a)x_0 = x$, we have

$$\|Px\| = \|P\pi(a)x_0\| = \|P\pi(a)P^{-1}x_0'\| = \|\pi\sigma(a)x_0'\| \leqslant \|\pi\sigma\|\|a\|\|x_0'\|.$$

Taking the infimum over all such a, we obtain $\|Px\| \leqslant \|\pi\sigma\|\|x_0'\|\|x\|$. Thus P is bounded and the closed graph theorem implies that P^{-1} is also bounded.

It remains to show that Q is bounded. Choose $a_0 \in \mathcal{A}$ such that $Qx_0 = \pi(a_0)x_0$ and set $\delta' = \delta - \mathrm{ad}_\sigma(a_0)$, where $\mathrm{ad}_\sigma(a_0)$ is the σ-derivation of $\mathcal{A}$ defined by $\mathrm{ad}_\sigma(a_0)(a) = \sigma(a)a_0 - a_0a$ for $a \in \mathcal{A}$. Then δ' is also a π-inner σ-derivation of $\mathcal{A}$. More precisely, $\pi\delta'(a) = \pi\sigma(a)Q' - Q'\pi(a)$ for all $a \in \mathcal{A}$, where $Q' = Q - \pi(a_0)$. Let $x \in X$. Then if $a \in \mathcal{A}$ with $\pi(a)x_0 = x$, we have

$$\|Q'x\| = \|Q'\pi(a)x_0\| = \|\pi\sigma(a)Q'x_0 - \pi\delta'(a)x_0\| = \|\pi\delta'(a)x_0\| \leqslant \|\pi\delta'\|\|a\|\|x_0\|,$$

since $Q'x_0 = 0$. Taking the infimum over all such a, we obtain $\|Q'x\| \leqslant \|\pi\delta'\|\|x_0\|\|x\|$. This implies that Q', and hence Q, is bounded. $\square$

Two automorphisms σ, τ of $\mathcal{A}$ are called *π-dependent* if $\sigma\tau^{-1}$ is π-inner, that is, there exists an invertible $P \in L(X)$ such that $\pi\sigma(a) = P\pi\tau(a)P^{-1}$ for all $a \in \mathcal{A}$. Otherwise, they are called *π-independent*. Clearly, an automorphism τ of $\mathcal{A}$ and the identity automorphism $1_{\mathcal{A}}$ are π-independent if and only if τ is π-outer. Also, it is easy to see that π-dependency is an equivalence relation for the automorphisms of $\mathcal{A}$.

Let σ be an automorphism of $\mathcal{A}$ and let $\delta_1, \ldots, \delta_n$ be σ-derivations of $\mathcal{A}$. We say that $\delta_1, \ldots, \delta_n$ are *$\mathbb{C}$-dependent modulo π-inner σ-derivations* if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^n \alpha_i\delta_i$ is π-inner. Otherwise, they are said to be *$\mathbb{C}$-independent modulo π-inner σ-derivations*. Clearly, a σ-derivation is $\mathbb{C}$-independent modulo π-inner σ-derivations if and only if it is π-outer .

These notions are generalized in [10] to σ-derivations of $\mathcal{A}$ into $L(X)$. Let σ be an automorphism of $\mathcal{A}$. By a σ-derivations δ' of $\mathcal{A}$ into $L(X)$ we mean a linear map $\delta' : \mathcal{A} \to L(X)$ such that $\delta'(ab) = \pi\sigma(a)\delta'(b) + \delta'(a)\pi(b)$ for all $a, b \in \mathcal{A}$. A σ-derivation δ' of $\mathcal{A}$ into $L(X)$ is called *inner* if there exists $Q \in L(X)$ such that $\delta'(a) = \pi\sigma(a)Q - Q\pi(a)$ for all $a \in A$. Otherwise, it is called *outer*. We say that the σ-derivations $\delta_1', \ldots, \delta_n{}'$ of $\mathcal{A}$ into $L(X)$ are *$\mathbb{C}$-dependent modulo inner σ-derivations* if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^n \alpha_i\delta_i{}'$ is inner. Otherwise, they are said to be *$\mathbb{C}$-independent modulo inner σ-derivations*.

We shall make use frequently of the following basic result concerning density properties of π-outer automorphisms and σ-derivations.

Theorem 2.2 [10, Theorem 3.4]. *Let σ be an automorphism of $\mathcal{A}$, $\tau_1, \ldots, \tau_m$ pairwise π-independent automorphisms of $\mathcal{A}$, and let $\delta_1{}', \ldots, \delta_n{}'$ be σ-derivations of $\mathcal{A}$ into $L(X)$ that are $\mathbb{C}$-independent modulo inner σ-derivations. Then for any $\mathbb{C}$-independent $x_1, \ldots, x_k \in X$ and arbitrary $y_{ij}, z_{\ell j} \in X$, $i = 1, \ldots, n$,*

$\ell = 1, \ldots, m$ and $j = 1, \ldots, k$, there exists $a \in \mathcal{A}$ such that $\delta_i{}'(a)x_j = y_{ij}$ and $\pi\tau_\ell(a)x_j = z_{\ell j}$ for all i, ℓ and j.

We remark that either m or n can be 0 in the preceding theorem. In the case that $m = 1$, the assumption of pairwise π-independence holds vacuously.

In what follows, we shall write $\widetilde{\theta} = \pi\theta : \mathcal{A} \to B(X)$ for any linear map θ on $\mathcal{A}$. Let σ be an automorphism of $\mathcal{A}$. If δ is a σ-derivation of $\mathcal{A}$, then $\widetilde{\delta}$ is a σ-derivation of $\mathcal{A}$ into $B(X)$. Clearly, the σ-derivations $\delta_1, \ldots, \delta_n$ of $\mathcal{A}$ are $\mathbb{C}$-dependent modulo π-inner σ-derivations if, and only if, the σ-derivations $\widetilde{\delta}_1, \ldots, \widetilde{\delta}_n$ of $\mathcal{A}$ into $B(X)$ are $\mathbb{C}$-dependent modulo inner σ-derivations. Let d, δ be σ-derivations of $\mathcal{A}$. For $a, b \in \mathcal{A}$, we have

$$d\delta(ab) = \sigma^2(a)d\delta(b) + d\sigma(a)\delta(b) + \sigma\delta(a)d(b) + d\delta(a)b,$$

and so

$$\widetilde{d\delta}(ab) = \widetilde{\sigma^2}(a)\widetilde{d\delta}(b) + \widetilde{d\sigma}(a)\widetilde{\delta}(b) + \widetilde{\sigma\delta}(a)\widetilde{d}(b) + \widetilde{d\delta}(a)\pi(b). \tag{2.1}$$

3. Reduction to d, δ being both π-inner

In this section, we shall show that the conclusion of Theorem 1.1 holds if at least one of d and δ is π-outer. We divide the situation into three cases.

Case 1. d and δ are both π-outer.

First we claim that d, δ are $\mathbb{C}$-dependent modulo π-inner σ-derivations. Assume on the contrary that they are $\mathbb{C}$-independent modulo π-inner σ-derivations and choose a nonzero $x \in X$. By Theorem 2.2, there exists $b \in \mathcal{A}$ such that $\widetilde{\delta}(b)x = x$, $\widetilde{d}(b)x = 0$ and $\pi(b)x = 0$. If x and $\widetilde{d\delta}(b)x$ are $\mathbb{C}$-independent, then, by Theorem 2.2 again, there exists $a' \in \mathcal{A}$ such that $\widetilde{d}(a')x = x$ and $\widetilde{\sigma}(a')\widetilde{d\delta}(b)x = 0$. But if $\widetilde{d\delta}(b)x = \beta x$ for some $\beta \in \mathbb{C}$, then it follows from the quasi-nilpotency of $\widetilde{d\delta}(b)$ that $\beta = 0$ and so we have also $\widetilde{d}(a')x = x$ and $\widetilde{\sigma}(a')\widetilde{d\delta}(b)x = 0$ for some $a' \in \mathcal{A}$. Let $a = \sigma^{-1}(a')$; then we have $\widetilde{d\delta}(ab)x = x$ by (2.1), contradicting the quasi-nilpotency of $\widetilde{d\delta}(ab)$. Therefore, there exist a nonzero $\alpha \in \mathbb{C}$ and $Q \in L(X)$ such that

$$\widetilde{d}(a) = \alpha\widetilde{\delta}(a) + \widetilde{\sigma}(a)Q - Q\pi(a)$$

for all $a \in \mathcal{A}$.

Next we show $\widetilde{d} = \alpha\widetilde{\delta}$. Suppose first that σ is π-outer. If $Q = 0$, we are done. So assume $Qx \neq 0$ for some $x \in X$. If Qx and x are $\mathbb{C}$-independent, then Theorem 2.2 guarantees the existence of $b \in \mathcal{A}$ such that $\widetilde{d}(b)x = 0$, $\widetilde{\sigma}(b)Qx \neq 0$ and $\pi(b)x = 0$. But if $Qx = \beta x$ for some nonzero $\beta \in \mathbb{C}$, we can pick $b \in \mathcal{A}$ so that $\widetilde{d}(b)x = 0$, $\widetilde{\sigma}(b)x \neq 0$ and $\pi(b)x = 0$. Then $\widetilde{\sigma}(b)Qx \neq 0$ either. Suppose next that σ is π-inner, that is, there exists an invertible $P \in L(X)$ such that $\widetilde{\sigma}(a) = P\pi(a)P^{-1}$ and so $\widetilde{d}(a) = \alpha\widetilde{\delta}(a) + P\pi(a)P^{-1}Q - Q\pi(a)$ for all $a \in \mathcal{A}$. If $P^{-1}Q \in \mathbb{C}I$, then $\widetilde{d} = \alpha\widetilde{\delta}$, as desired. So assume $P^{-1}Q \notin \mathbb{C}I$; then $P^{-1}Qx$ and x are $\mathbb{C}$-independent for some $x \in X$. By Theorem 2.2, there exists $b \in \mathcal{A}$ such that $\widetilde{d}(b)x = 0$, $\pi(b)x = 0$ and $\pi(b)P^{-1}Qx \neq 0$. Hence $\widetilde{\sigma}(b)Qx = P\pi(b)P^{-1}Qx \neq 0$ either. Thus we have shown that if $\widetilde{d} \neq \alpha\widetilde{\delta}$, there exist $x \in X$ and $b \in \mathcal{A}$ such that $\widetilde{d}(b)x = 0$, $\pi(b)x = 0$ and $\widetilde{\sigma}(b)Qx \neq 0$. Then $\alpha\widetilde{\delta}(b)x = \widetilde{d}(b)x - (\widetilde{\sigma}(b)Q - Q\pi(b))x = -\widetilde{\sigma}(b)Qx \neq 0$ and so $\widetilde{\delta}(b)x \neq 0$. If $\widetilde{d\delta}(b)x$ and $\widetilde{\delta}(b)x$ are $\mathbb{C}$-independent, by Theorem 2.2, we can choose $a' \in \mathcal{A}$ such that $\widetilde{\sigma}(a')\widetilde{d\delta}(b)x = 0$ and $\widetilde{d}(a')\widetilde{\delta}(b)x = x$. Obviously we can also do so even if $\widetilde{d\delta}(b)x$ and $\widetilde{\delta}(b)x$ are $\mathbb{C}$-dependent. Let $a = \sigma^{-1}(a')$; then $\widetilde{d\delta}(ab)x = x$ by (2.1), contradicting the quasi-nilpotency of $\widetilde{d\delta}(ab)$. Consequently, we have $\widetilde{d} = \alpha\widetilde{\delta}$.

Using $\widetilde{d} = \alpha\widetilde{\delta}$ and $\sigma\delta = q\delta\sigma$, we obtain from (2.1)

$$\widetilde{d\delta}(ab) = \alpha\widetilde{\sigma^2}(a)\widetilde{\delta^2}(b) + \alpha(1+q)\widetilde{\delta\sigma}(a)\widetilde{\delta}(b) + \alpha\widetilde{\delta^2}(a)\pi(b) \tag{3.1}$$

for all $a, b \in \mathcal{A}$. Let $x \in X$ and $x \neq 0$. By Theorem 2.2, there exists $b \in \mathcal{A}$ such that $\widetilde{\delta}(b)x = x$ and $\pi(b)x = 0$. Also, there exists $a' \in \mathcal{A}$ such that $\widetilde{\delta}(a')x = x$ and $\widetilde{\sigma}(a')\widetilde{\delta^2}(b)x = 0$. Let $a = \sigma^{-1}(a')$;

then $\widetilde{d\delta}(ab)x = \alpha(1+q)x$, implying that $1+q=0$. Then it follows from (3.1) that $\widetilde{d\delta}(ab) = \alpha\sigma^2(a)\widetilde{\delta^2}(b) + \alpha\widetilde{\delta^2}(a)\pi(b) = \widetilde{\sigma^2}(a)\widetilde{d\delta}(b) + \widetilde{d\delta}(a)\pi(b)$ for all $a, b \in \mathcal{A}$, and so $\widetilde{d\delta}$ is a σ^2-derivation of $\mathcal{A}$ into $B(X)$.

Moreover, $\widetilde{d\delta}$ must be inner; otherwise, for any nonzero $x \in X$, there would exist $a \in \mathcal{A}$ such that $\widetilde{d\delta}(a)x = x$, a contradiction. Let $Q \in L(X)$ such that $\widetilde{d\delta}(a) = \widetilde{\sigma^2}(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. If $Q = 0$, then $\widetilde{d\delta} = 0$ and we are done. Assume $Qx \neq 0$ for some $x \in X$. Then σ^2 is π-inner; otherwise there would exist $a \in \mathcal{A}$ such that $\widetilde{\sigma^2}(a)Qx = x$ and $\pi(a)x = 0$, and so $\widetilde{d\delta}(a)x = x$, a contradiction. Hence there exists an invertible $P \in L(X)$ such that $\widetilde{\sigma^2}(a) = P\pi(a)P^{-1}$ and so $\widetilde{d\delta}(a) = P\pi(a)P^{-1}Q - Q\pi(a)$ for all $a \in \mathcal{A}$. If $P^{-1}Q \notin \mathbb{C}I$, there would exist $x \in X$ such that $P^{-1}Qx, x$ are $\mathbb{C}$-independent, and $a \in \mathcal{A}$ such that $\pi(a)x = 0$ and $\pi(a)P^{-1}Qx = P^{-1}x$, and then $\widetilde{d\delta}(a)x = x$, a contradiction. Hence, $P^{-1}Q \in \mathbb{C}I$. Thus $\widetilde{d\delta} = 0$ and we are done.

Case 2. d is π-outer and δ is π-inner.

Let $Q \in L(X)$ such that $\widetilde{\delta}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. We claim $\widetilde{\delta} = 0$ in this case. Assume on the contrary $\widetilde{\delta} \neq 0$; then $Q \neq 0$.

Suppose first that σ is π-outer. Let $x \in X$ with $Qx \neq 0$ and choose $b \in \mathcal{A}$ such that $\widetilde{\sigma}(b)Qx = x$, $\pi(b)x = 0$ and $\widetilde{d}(b)x = 0$. Suppose next that σ is π-inner, say, there exists an invertible $P \in L(X)$ such that $\widetilde{\sigma}(a) = P\pi(a)P^{-1}$ and so $\widetilde{\delta}(a) = P\pi(a)P^{-1}Q - Q\pi(a)$ for all $a \in \mathcal{A}$. Note that $P^{-1}Q \notin \mathbb{C}I$ since $\widetilde{\delta} \neq 0$, so we can choose $x \in X$ such that $P^{-1}Qx, x$ are $\mathbb{C}$-independent. Let $b \in \mathcal{A}$ such that $\pi(b)P^{-1}Qx = P^{-1}x$, $\pi(b)x = 0$ and $\widetilde{d}(b)x = 0$; then $\widetilde{\sigma}(b)Qx = x$ holds too. In either situation, we have $\widetilde{\sigma}(b)Qx = x$, $\pi(b)x = 0$ and $\widetilde{d}(b)x = 0$ for some $b \in \mathcal{A}$ and nonzero $x \in X$. Then $\widetilde{\delta}(b)x = x$. Let $a' \in \mathcal{A}$ such that $\widetilde{d}(a')x = x$ and $\widetilde{\sigma}(a')\widetilde{d\delta}(b)x = 0$, and set $a = \sigma^{-1}(a')$; then $\widetilde{d\delta}(ab)x = x$ by (2.1), a contradiction.

Therefore $\widetilde{\delta} = 0$. Then $\widetilde{\sigma\delta} = q\widetilde{\delta\sigma} = 0$ and so it follows from (2.1) that $\widetilde{d\delta}(ab) = \widetilde{\sigma^2}(a)\widetilde{d\delta}(b) + \widetilde{d\delta}(a)\pi(b)$ for all $a, b \in \mathcal{A}$. Thus $\widetilde{d\delta}$ is a σ^2-derivation of $\mathcal{A}$ into $L(X)$, and hence $\widetilde{d\delta} = 0$ as we have shown in Case 1.

Case 3. d is π-inner and δ is π-outer.

Let $Q \in L(X)$ such that $\widetilde{d}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. Assuming $\widetilde{d} \neq 0$ and proceeding as in Case 2 with d and δ exchanged, we can choose a nonzero $x \in X$ and $b \in \mathcal{A}$ such that $\widetilde{\sigma}(b)Qx = x$, $\pi(b)x = 0$ and $\widetilde{\delta}(b)x = 0$. Then $\widetilde{d}(b)x = x$. Let $a' \in \mathcal{A}$ such that $\widetilde{\delta}(a')x = x$ and $\widetilde{\sigma}(a')\widetilde{d\delta}(b)x = 0$, and set $a = \sigma^{-1}(a')$; then $\widetilde{\sigma\delta}(a)x = q\widetilde{\delta\sigma}(a)x = q\widetilde{\delta}(a')x = qx$ and hence $\widetilde{d\delta}(ab)x = qx$ by (2.1), a contradiction. Hence $\widetilde{d} = 0$. Then $\widetilde{d\delta} = 0$ and we are done.

4. Reduction to σ being π-inner

Thus we may assume henceforth that both d and δ are π-inner. In this section, we shall show that we may assume further that σ is also π-inner. We assume on the contrary that σ is π-outer, and proceed according as whether σ^2 is π-outer. Write $\widetilde{d}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ and $\widetilde{\delta}(a) = \widetilde{\sigma}(a)R - R\pi(a)$ for all $a \in \mathcal{A}$, where $Q, R \in L(X)$. Then $\widetilde{d\delta}(a) = \widetilde{\sigma\delta}(a)Q - Q\pi(\delta(a)) = q\widetilde{\delta}(\sigma(a))Q - Q\widetilde{\delta}(a)$, and so

$$\widetilde{d\delta}(a) = q\widetilde{\sigma^2}(a)RQ - qR\widetilde{\sigma}(a)Q - Q\widetilde{\sigma}(a)R + QR\pi(a) \tag{4.1}$$

for all $a \in \mathcal{A}$.

Case 1. σ^2 is π-outer.

Note that $1_{\mathcal{A}}, \sigma, \sigma^2$ are pairwise π-independent in this case. First we have $RQ = 0$; for if $RQx \neq 0$ for some $x \in X$, then by Theorem 2.2, there would exist $a \in \mathcal{A}$ such that $\widetilde{\sigma^2}(a)RQx = x$ and $\widetilde{\sigma}(a)Qx = \widetilde{\sigma}(a)Rx = \pi(a)x = 0$, and so $\widetilde{d\delta}(a)x = qx$ by (4.1), a contradiction. Similarly $QR = 0$; for if $QRx \neq 0$ for some $x \in X$, then by Theorem 2.2, there would exist $a \in \mathcal{A}$ such that $\pi(a)QRx = x$ and $\widetilde{\sigma}(a)Q^2Rx = 0$, and so $\widetilde{d\delta}(a)QRx = QRx$, a contradiction. Thus (4.1) is reduced to

$$\widetilde{d\delta}(a) = -qR\widetilde{\sigma}(a)Q - Q\widetilde{\sigma}(a)R \tag{4.2}$$

for all $a \in \mathcal{A}$.

Now we claim either $Q^2 = 0$ or $R^2 = 0$. Assume on the contrary $Q^2 \neq 0$ and $R^2 \neq 0$. Then we can choose $x \in X$ such that $Q^2x \neq 0$ and $R^2x \neq 0$; otherwise the vector space X would be the set-theoretic union of two proper subspaces, namely the kernels of Q^2 and R^2. From $QR = RQ = 0$, it follows that Qx and Rx are $\mathbb{C}$-independent. If Q^2x and R^2x were also $\mathbb{C}$-independent, there would exist $a \in \mathcal{A}$ such that $\widetilde{\sigma}(a)Q^2x = \frac{1}{q}x$ and $\widetilde{\sigma}(a)R^2x = x$, and so $\widetilde{d\delta}(a)(Qx + Rx) = -(Qx + Rx)$, a contradiction. Hence $R^2x = \alpha Q^2x$ for some nonzero $\alpha \in \mathbb{C}$. Let $\gamma \in \mathbb{C}$ such that $\gamma^2 = \frac{q}{\alpha}$, and $a \in \mathcal{A}$ such that $\widetilde{\sigma}(a)Q^2x = x$; then $\widetilde{d\delta}(a)(Qx + \gamma Rx) = -\alpha\gamma(Qx + \gamma Rx)$, a contradiction again. Hence either $Q^2 = 0$ or $R^2 = 0$. Thus $\widetilde{d\delta}(a)^3 = 0$ for all $a \in \mathcal{A}$ by (4.2) and so we are done.

Case 2. σ^2 is π-inner.

Let $P \in L(X)$ be invertible such that $\widetilde{\sigma^2}(a) = P\pi(a)P^{-1}$ for all $a \in \mathcal{A}$. Then (4.1) can be rewritten as

$$\widetilde{d\delta}(a) = qP\pi(a)P^{-1}RQ - qR\widetilde{\sigma}(a)Q - Q\widetilde{\sigma}(a)R + QR\pi(a)$$

for all $a \in \mathcal{A}$. If $P^{-1}RQx$ and x were $\mathbb{C}$-independent for some $x \in X$, then by Theorem 2.2, there would exist $a \in \mathcal{A}$ such that $\widetilde{\sigma}(a)Qx = \widetilde{\sigma}(a)Ux = 0$, $\pi(a)x = 0$ and $\pi(a)P^{-1}RQx = P^{-1}x$, and so $\widetilde{d\delta}(a)x = qx$, a contradiction. Hence $P^{-1}RQ \in \mathbb{C}I$ and then

$$\widetilde{d\delta}(a) = (qRQ + QR)\pi(a) - qR\widetilde{\sigma}(a)Q - Q\widetilde{\sigma}(a)R$$

for all $a \in \mathcal{A}$. If $y = (qRQ + QR)x \neq 0$ for some $x \in X$, by Theorem 2.2, there would exist $a \in \mathcal{A}$ such that $\pi(a)y = x$ and $\widetilde{\sigma}(a)Qy = \widetilde{\sigma}(a)Ry = 0$, and so $\widetilde{d\delta}(a)y = y$, a contradiction. Hence $qRQ + QR = 0$ and then we have (4.2) again. Thus if $RQ = 0$, then $QR = 0$ and so we are done as in Case 1. Assume $RQ \neq 0$. Then both R and Q are invertible in $L(X)$ since $P^{-1}RQ \in \mathbb{C}I$. If $Q^{-1}Rx$, x were $\mathbb{C}$-independent for some $x \in X$, then so were Qx and Rx, and there would exist $a \in \mathcal{A}$ such that $\widetilde{\sigma}(a)Qx = R^{-1}x$ and $\widetilde{\sigma}(a)Rx = 0$, and so $\widetilde{d\delta}(a)x = -qx$ by (4.2), a contradiction. Hence $Q^{-1}R = \alpha I$, or equivalently $R = \alpha Q$ for some nonzero $\alpha \in \mathbb{C}$. Thus $\widetilde{d\delta}(a) = -\alpha(q + 1)Q\widetilde{\sigma}(a)Q$ for all $a \in \mathcal{A}$. Choose a nonzero $x \in X$ and $a \in \mathcal{A}$ such that $\widetilde{\sigma}(a)Qx = Q^{-1}x$. Then $\widetilde{d\delta}(a)x = -\alpha(q + 1)x$, implying that $\alpha(q + 1) = 0$. So $\widetilde{d\delta} = 0$ and we are done.

5. The case d, δ and σ being all π-inner

Now we may assume that d, δ and σ are all π-inner. Thus there exist $P, Q, R \in B(X)$ with P invertible such that $\widetilde{\sigma}(a) = P\pi(a)P^{-1}$, $\widetilde{d}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ and $\widetilde{\delta}(a) = \widetilde{\sigma}(a)R - R\pi(a)$ for all $a \in \mathcal{A}$. We may choose the maps P, Q and R so that Q and R are "q-skew"-commuting with P in the sense of the following proposition.

Proposition 5.1. *Let σ be an automorphism of $\mathcal{A}$ and let d, δ be q-skew σ-derivations of $\mathcal{A}$ for some nonzero $q \in \mathbb{C}$. Suppose that σ, d, δ are all π-inner. Then there exist $P, Q, R \in B(X)$ with P invertible in $B(X)$ such that $PQP^{-1} = qQ$, $PRP^{-1} = qR$, $\widetilde{\sigma}(a) = P\pi(a)P^{-1}$, $\widetilde{d}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ and $\widetilde{\delta}(a) = \widetilde{\sigma}(a)R - R\pi(a)$ for all $a \in \mathcal{A}$.*

Proof. By Proposition 2.1, there exist $P, Q \in B(X)$ with P invertible in $B(X)$ such that $\pi\sigma(a) = P\pi(a)P^{-1}$ and $\pi d(a) = \pi\sigma(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. Then $\pi\sigma^{-1}(a) = P^{-1}\pi(a)P$ for all $a \in \mathcal{A}$. By assumption, we have $\pi\sigma d\sigma^{-1}(a) = q\pi d(a)$ for all $a \in \mathcal{A}$. On the one hand, $\pi\sigma d\sigma^{-1}(a) = P\pi d\sigma^{-1}(a)P^{-1} = P(\pi(a)Q - Q\pi\sigma^{-1}(a))P^{-1} = P(\pi(a)Q - QP^{-1}\pi(a)P)P^{-1} = P(\pi(a)QP^{-1} - QP^{-1}\pi(a))$. On the other hand, $q\pi d(a) = q(\pi\sigma(a)Q - Q\pi(a)) = q(P\pi(a)P^{-1}Q - Q\pi(a)) = qP(\pi(a)P^{-1}Q - P^{-1}Q\pi(a))$. Comparing these two identities, we obtain $\pi(a)(QP^{-1} - qP^{-1}Q) = (QP^{-1} - qP^{-1}Q)\pi(a)$ for all $a \in \mathcal{A}$. Since $\pi(\mathcal{A})$ acts densely on X, it follows that $QP^{-1} - qP^{-1}Q = \lambda I \in \mathbb{C}I$.

If $\lambda = 0$, then $PQP^{-1} = qQ$ as desired. Hence we may assume that $\lambda \neq 0$. We claim that $q \neq 1$ in this case. Assume on the contrary that $q = 1$; then $[Q, P^{-1}] = \lambda I$ and so $[[Q, P^{-1}], P^{-1}] = 0$. Since $Q, P^{-1} \in B(X)$ which is a Banach algebra, it follows from Kleinecke–Shirokov Theorem [16,21] that $[Q, P^{-1}] = \lambda I$ is quasi-nilpotent, a contradiction. Hence $q \neq 1$.

From $PQP^{-1} - qQ = \lambda P$, it follows that $P_1QP_1^{-1} - qQ = P_1$, where $P_1 = \lambda P$. Let $Q_1 = Q + \frac{P_1}{q-1}$; then

$$P_1Q_1P_1^{-1} = P_1QP_1^{-1} + \frac{P_1}{q-1} = qQ + P_1 + \frac{P_1}{q-1} = q\left(Q + \frac{P_1}{q-1}\right) = qQ_1.$$

Moreover, it is easy to see that $\widetilde{\sigma}(a) = P_1\pi(a)P_1^{-1}$ and $\widetilde{d}(a) = \widetilde{\sigma}(a)Q_1 - Q_1\pi(a)$ for all $a \in \mathcal{A}$. Replacing the maps P and Q by P_1 and Q_1, respectively, we may assume $PQP^{-1} = qQ$ such that $\widetilde{\sigma}(a) = P\pi(a)P^{-1}$ and $\widetilde{d}(a) = \widetilde{\sigma}(a)Q - Q\pi(a)$ for all $a \in \mathcal{A}$. Let $R \in B(X)$ such that $\widetilde{\delta}(a) = \widetilde{\sigma}(a)R - R\pi(a)$ for all $a \in \mathcal{A}$; then $RP^{-1} - qP^{-1}R = \mu I \in \mathbb{C}I$ as we have seen above. If $\mu = 0$, we are done. Otherwise, $q \neq 1$. Replacing the maps P and R by μP and $R + \frac{\mu P}{q-1}$, respectively, we are done too. □

Let $P, Q, R \in B(X)$ be as described in Proposition 5.1. Set $A = P^{-1}Q$ and $B = P^{-1}R$; then $PA = qAP$ and $PB = qBP$. Thus, $\widetilde{d}(a) = P\pi(a)A - PA\pi(a) = P[\pi(a), A]$, $\widetilde{\delta}(a) = P[\pi(a), B]$ and so $\widetilde{d\delta}(a) = P[\widetilde{\delta}(a), A] = P[P[\pi(a), B], A]$ for all $a \in \mathcal{A}$. Then Theorem 1.1 follows immediately from the following purely algebraic result.

Theorem 5.2. *Let X be a complex vector space and let $\mathcal{B}$ be a dense algebra of linear operators on X. Let A, B, P be linear operators on X with P invertible such that $PA = qAP$, $PB = qBP$ for some nonzero $q \in \mathbb{C}$. If $H(T) = P[P[T, B], A]$ is quasi-nilpotent for every $T \in \mathcal{B}$, then $H(T)^3 = 0$ for all $T \in \mathcal{B}$.*

Since $PA = qAP$, $PB = qBP$, an expansion of $H(T)$ yields

$$\begin{aligned} H(T) &= P^2TBA - P^2BTA - PAPTB + PAPBT \\ &= (P^2T)BA - q^2B(P^2T)A - qA(P^2T)B + q^3AB(P^2T). \end{aligned}$$

Thus the hypothesis of Theorem 5.2 can be reformulated as that

$$H_1(T) = TBA - q^2BTA - qATB + q^3ABT \tag{5.1}$$

is quasi-nilpotent for every $T \in \mathcal{B}_1$, where $\mathcal{B}_1 = P^2\mathcal{B}$ is a dense subset of $L(X)$. And, the conclusion $H(T)^3 = 0$ for all $T \in \mathcal{B}$ is just $H_1(T)^3 = 0$ for all $T \in \mathcal{B}_1$. We begin with an interesting consequence of the quasi-nilpotency of $H_1(T)$.

Lemma 5.3. *$BAx \in \operatorname{span}\{Ax, Bx, x\}$ for every $x \in X$. Moreover, for every $x \in X$, there exist $\mu, \nu \in \mathbb{C}$ such that $BAx = \mu Ax + \nu Bx - \mu\nu x$.*

Proof. Assume on the contrary $BAx \notin \operatorname{span}\{Ax, Bx, x\}$ for some $x \in X$. By the density of $\mathcal{B}_1$ on X, there exists $T \in \mathcal{B}_1$ such that $TBAx = x$ and $TAx = TBx = Tx = 0$. Then $H_1(T)x = x$ by (5.1), contradicting the quasi-nilpotency of $H_1(T)$. Hence, $BAx \in \operatorname{span}\{Ax, Bx, x\}$ for every $x \in X$.

Thus, for each $x \in X$, we may write $BAx = \mu Ax + \nu Bx + \lambda x$ for some $\mu, \nu, \lambda \in \mathbb{C}$. Then it follows from (5.1) that

$$H_1(T)x = (\mu I - q^2B)TAx + (\nu I - qA)TBx + (\lambda I + q^3AB)Tx$$

for all $T \in \mathcal{B}_1$. We proceed to show that μ, ν, λ can be chosen so that $\lambda = -\mu\nu$. Suppose first that Ax, Bx and x are $\mathbb{C}$-independent. Let $T \in \mathcal{B}_1$ such that $TAx = \nu x$, $TBx = q^2Bx$ and $Tx = x$; then $H_1(T)x = (\lambda + \mu\nu)x$ and so $\lambda = -\mu\nu$, as desired.

Suppose next that Ax, Bx and x are $\mathbb{C}$-dependent. If Ax and x are $\mathbb{C}$-dependent, say $Ax = \alpha x$ for some $\alpha \in \mathbb{C}$, then $BAx = \alpha Bx$ and we are done by taking $\mu = 0$ and $\nu = \alpha$.

So we assume that Ax and x are $\mathbb{C}$-independent. Suppose first that Bx and x are $\mathbb{C}$-dependent, say, $Bx = \beta x$ for some $\beta \in \mathbb{C}$. Then by assumption we may write $BAx = \mu Ax + \lambda x$ for some $\mu, \lambda \in \mathbb{C}$

and $ABx = \beta Ax$. So by (5.1)

$$H_1(T)x = (\mu I - q^2B)TAx + (-\beta qA + \lambda I + q^3AB)Tx$$

for all $T \in \mathcal{B}_1$. We claim $\lambda = 0$, whence $BAx = \mu Ax$ and so $\nu = 0$ works. Assume on the contrary $\lambda \neq 0$. Let $T \in \mathcal{B}_1$ such that $TAx = x$ and $Tx = 0$; then $H_1(T)x = (\mu - \beta q^2)x$, implying $\mu = \beta q^2$. Thus $BAx = \beta q^2Ax + \lambda x$ and

$$H_1(T)x = q^2(\beta I - B)TAx + (-\beta qA + \lambda I + q^3AB)Tx$$

for all $T \in \mathcal{B}_1$. Now, let $T \in \mathcal{B}_1$ such that $TAx = Ax$ and $Tx = qx$; then $H_1(T)x = \lambda q(1 - q)x$, implying $q = 1$. Thus $BAx = \beta Ax + \lambda x$ and

$$H_1(T)x = (\beta I - B)TAx + (-\beta A + \lambda I + AB)Tx$$

for all $T \in \mathcal{B}_1$. Finally, let $T \in \mathcal{B}_1$ such that $TAx = 0$ and $Tx = x$; then $H_1(T)x = \lambda x$, a contradiction. Hence, $\lambda = 0$ and we are done.

Suppose next that Bx and x are $\mathbb{C}$-independent. Write $Bx = \alpha Ax + \beta x$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Thus $BAx = \gamma Ax + \theta x$ for some $\gamma, \theta \in \mathbb{C}$. Let $\nu \in \mathbb{C}$ be a solution of the quadratic equation $\alpha\nu^2 + (\beta - \gamma)\nu - \theta = 0$ and set $\mu = \gamma - \alpha\nu$. Then

$$\begin{aligned} BAx &= (\mu + \alpha\nu)Ax + (\alpha\nu^2 + (\beta - \gamma)\nu)x = \mu Ax + \nu(\alpha Ax + \beta x) - (\gamma - \alpha\nu)\nu x \\ &= \mu Ax + \nu Bx - \mu\nu x \end{aligned}$$

as required. □

Lemma 5.4. *Either $H(T)^3 = 0$ for all $T \in \mathcal{B}$ or there exists $y \in X$ such that Ay, By and y are $\mathbb{C}$-independent.*

Proof. Suppose that Ax, Bx and x are $\mathbb{C}$-dependent for every $x \in X$. We are going to show that $H(T)^3 = 0$ for all $T \in \mathcal{B}$. According to [5, Theorem 2.4], we have three cases to discuss.

Case 1. A, B and I are $\mathbb{C}$-dependent.

If $A \in \mathbb{C}I$ or $B \in \mathbb{C}I$, then $H(T) = P[P[T, B], A] = 0$ for all $T \in \mathcal{B}$ and we are done. So we assume that $B = \alpha A + \beta I$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Then $A^2x \in \operatorname{span}\{BAx, Ax\} \subseteq \operatorname{span}\{Ax, Bx, x\} \subseteq \operatorname{span}\{Ax, x\}$ by Lemma 5.3 and so A^2x, Ax and x are $\mathbb{C}$-dependent for every $x \in X$. By Kaplansky's Theorem [15, Lemma 14], A is algebraic of degree at most 2. So $(A - \gamma_1 I)(A - \gamma_2 I) = 0$ for some $\gamma_1, \gamma_2 \in \mathbb{C}$. Replacing A by $A - \lambda_1 I$, we may assume $A(A - \gamma I) = 0$ for some $\gamma \in \mathbb{C}$. Then

$$\begin{aligned} H(T) &= P[P[T, B], A] = P[P[T, \alpha A], A] \\ &= \alpha(P^2TA^2 - P^2ATA - PAPTA + PAPAT) \\ &= \alpha(\gamma P^2 - P^2A - PAP)TA + \alpha PAPAT = MTA + NT \end{aligned}$$

for all $T \in \mathcal{B}$, where $M = \alpha(\gamma P^2 - P^2A - PAP)$ and $N = \alpha PAPA$. If ANx and Nx were $\mathbb{C}$-independent for some $x \in X$, there would exist $T \in \mathcal{B}$ such that $TANx = 0$ and $TNx = x$, and so $H(T)Nx = Nx$, a contradiction. Hence $ANx \in \mathbb{C}Nx$ for every $x \in X$. By [7, Lemma 2.7] $AN = \lambda_1 N$ for some $\lambda_1 \in \mathbb{C}$. Similarly, $AM = \lambda_2 M$ and $A(M + N) = \lambda_3(M + N)$ for some $\lambda_2, \lambda_3 \in \mathbb{C}$. Assume that M and N are $\mathbb{C}$-independent. Then $\lambda_1 = \lambda_2 = \lambda_3$, so we may write $AN = \lambda N$ and $AM = \lambda M$ for some $\lambda \in \mathbb{C}$. Let $x \in X$ such that $(N + \lambda M)x \neq 0$ and let $T \in \mathcal{B}$ such that $T(N + \lambda M)x = x$. Then $H(T)(N + \lambda M)x = (N + \lambda M)x$, a contradiction. Hence $N = \theta M$ for some $\theta \in \mathbb{C}$; then $H(T) = MT(A + \theta I)$ for all $T \in \mathcal{B}$. If $(A + \theta I)Mx \neq 0$ for some $x \in X$, there would exist $T \in \mathcal{B}$ such that $T(A + \theta I)Mx = x$, and so $H(T)Mx = Mx$, a contradiction. Consequently, $(A + \theta I)M = 0$ and then $H(T)^2 = 0$ for all $T \in \mathcal{B}$, as desired.

Case 2. $\dim X \leqslant 3$.

In this case, $L(X) \cong M_3(\mathbb{C})$, the 3×3 matrix algebra over $\mathbb{C}$. So $H(T)$ is nilpotent and then $H(T)^3 = 0$ for all $T \in \mathcal{B}$.

Case 3. There is an idempotent $E \in L(X)$ of rank one such that $(I - E)A = \alpha(I - E)$, where $\alpha \in \mathbb{C}$. Then $A - \alpha I = E(A - \alpha I)$ has rank at most one. From

$$H(T) = P[P[T, B], A] = P[P[T, B], A - \alpha I] = P^2[T, B](A - \alpha I) - P(A - \alpha I)P[T, B],$$

it follows that $H(T)$ has rank at most two. As $H(T)$ is quasi-nilpotent, it follows that $H(T)^3 = 0$ for all $T \in \mathcal{B}$. $\square$

In view of Lemma 5.4, we may assume in what follows that there exists $y \in X$ such that Ay, By and y are $\mathbb{C}$-independent. Henceforth, we shall fix the notation y for this specified vector. By Lemma 5.3, there exist $\mu, \nu \in \mathbb{C}$ such that $BAy = \mu Ay + \nu By - \mu\nu y$. We shall also keep the notations μ, ν fixed. Set $A' = A - \nu I$ and $B' = B - \mu I$; then $B'A'y = 0$. Moreover, $A'y$, $B'y$ and y are also $\mathbb{C}$-independent and $\mathrm{span}\{A'y, B'y, y\} = \mathrm{span}\{Ay, By, y\}$. Note that it follows from (5.1) that

$$\begin{aligned} H_1(T) = {} & TB'A' - (q^2B' + \mu(q+1)(q-1)I)TA' - (qA' + \nu(q-1)I)TB' \\ & + (q^3A'B' + \mu q(q+1)(q-1)A' + \nu q^2(q-1)B' + \mu\nu(q+1)(q-1)^2I)T \end{aligned} \tag{5.2}$$

for all $T \in \mathcal{B}_1$.

Lemma 5.5. *Suppose that there exists $x \in X$ such that $B'x = 0$ but $B'A'x \neq 0$. Then* (1) $\nu = 0$, (2) $B'A'z = \mu(q+1)(q-1)A'z$ *for all $z \in X$ with $B'z = 0$,* (3) $A'^2x = 0$.

Proof. By Lemma 5.3, $BAx = \mu_x Ax + \nu_x Bx - \mu_x\nu_x x$, or equivalently, $(B - \mu_x I)(A - \nu_x I)x = 0$ for some $\mu_x, \nu_x \in \mathbb{C}$. Then $(B' - (\mu_x - \mu)I)(A' - (\nu_x - \nu)I)x = 0$ and the expansion of this yields $B'A'x = \alpha(A'x + \beta x)$, where $\alpha = \mu_x - \mu$ and $\beta = -(\nu_x - \nu)$. Hence, $\alpha \neq 0$.

Since $B'x = 0$ but $B'A'x \neq 0$, $A'x$ and x are $\mathbb{C}$-independent. Let $T \in \mathcal{B}_1$ such that $TA'x = x$ and $Tx = 0$; then $H_1(T)x = (\alpha - \mu(q+1)(q-1))x$ by (5.2) and so $\alpha = \mu(q+1)(q-1)$. Thus (5.2) can be rewritten as

$$\begin{aligned} H_1(T) = {} & TB'A' - (q^2B' + \alpha I)TA' - (qA' + \nu(q-1)I)TB' \\ & + (q^3A'B' + \alpha qA' + \nu q^2(q-1)B' + \nu\alpha(q-1)I)T. \end{aligned} \tag{5.3}$$

Next, let $T \in \mathcal{B}_1$ such that $TA'x = A'x$ and $Tx = qx$; then $H_1(T)x = \alpha(\nu - \beta)q(q-1)x$ and so $\beta = \nu$ since $\alpha = \mu(q+1)(q-1) \neq 0$. Thus we have $B'A'x = \alpha(A'x + \nu x)$. Now let $T \in \mathcal{B}_1$ such that $TA'y = -qA'y$, $TB'y = \alpha y$ and $Ty = 0$. Then $H_1(T)y = -\alpha\nu(q-1)y$ and so $\alpha\nu(q-1) = 0$. Thus we have (1) $\nu = 0$. Hence $B'A'x = \alpha A'x$ and (5.3) is reduced to

$$H_1(T) = TB'A' - (q^2B' + \alpha I)TA' - qA'TB' + (q^3A'B' + \alpha qA')T. \tag{5.4}$$

Let $z \in X$ with $B'z = 0$. If $B'A'z \neq 0$, we have also $B'A'z = \alpha A'z$ as shown above with z in place of x. Suppose $B'A'z = 0$; then $B'(x + z) = 0$ but $B'A'(x + z) = B'A'x \neq 0$, and so $B'A'(x + z) = \alpha A'(x + z)$ whence $B'A'z = \alpha A'z$. In either case, we have (2) $B'A'z = \mu(q+1)(q-1)A'z$ for all $z \in X$ with $B'z = 0$.

It remains to show that $A'^2x = 0$. Note that $A'x \neq 0$ since $B'A'x \neq 0$. Suppose first that A'^2x and $A'x$ are $\mathbb{C}$-dependent and write $A'^2x = \theta A'x$ for some $\theta \in \mathbb{C}$. Let $T \in \mathcal{B}_1$ such that $TA'x = A'x$. Then $TA'^2x = \theta A'x$, $TB'A'x = \alpha A'x$ and $TB'A'^2x = \alpha\theta A'x$. From (5.4) it follows that $H_1(T)A'x = \alpha\theta q^2(q-1)A'x$. So $\theta = 0$ and then $A'^2x = 0$, as desired. Suppose next that A'^2x and $A'x$ are $\mathbb{C}$-independent.

Then $B'A'^2x \in \text{span}\{A'^2x, A'x\}$; for otherwise there would exist $T \in \mathcal{B}_1$ such that $TB'A'^2x = A'x$ and $TA'^2x = TA'x = 0$, and so $H_1(T)A'x = A'x$ by (5.4), a contradiction. Hence we may write $B'A'^2x = \ell A'^2x + kA'x$ for some $\ell, k \in \mathbb{C}$. Let $T \in \mathcal{B}_1$ such that $TA'^2x = A'x$ and $TA'x = 0$. Then $TB'A'x = 0$ and $H_1(T)A'x = (\ell - \alpha(1+q^2))A'x$ by (5.4). Thus $\ell = \alpha(1+q^2)$.

If $A'^2x, A'x, x$ were $\mathbb{C}$-independent, then there would exist $T \in \mathcal{B}_1$ such that $TA'^2x = x$, $TA'x = 0$ and $Tx = qx$; then $TB'A'(A'x + x) = \ell x = \alpha(1+q^2)x$, $TA'(A'x+x) = x$, $TB'(A'x+x) = 0$ and $T(A'x+x) = qx$, and so $H_1(T)(A'x+x) = \alpha q^2(A'x+x)$ by (5.4), a contradiction. Hence, we may write $A'^2x = mA'x + nx$, where $m, n \in \mathbb{C}$. Then $B'A'^2x = \alpha mA'x = \ell A'^2x + kA'x$ and so the $\mathbb{C}$-independence of A'^2x and $A'x$ implies that $\ell = \alpha(1+q^2) = 0$ and $k = \alpha m$. Thus $q^2 = -1$. Let $T \in \mathcal{B}_1$ such that $TA'^2x = -qnx$ and $TA'x = A'x$. Then $TB'A'^2x = \alpha mA'x$ and $TB'A'x = \alpha A'x$, and so $H_1(T)A'x = \alpha m(1-q)A'x$ by (5.4). Thus $m = 0$ and so $B'A'^2x = 0$ and $A'^2x = nx$. Let $\gamma \in \mathbb{C}$ with $\gamma^2 = -n/q$. Let $T \in \mathcal{B}_1$ such that $TA'x = 0$ and $Tx = x$. Then $TB'A'(A'x+\gamma x) = TB'(A'x+\gamma x) = 0$, $TA'(A'x+\gamma x) = nx$ and $T(A'x+\gamma x) = \gamma x$. Thus $H_1(T)(A'x+\gamma x) = \alpha\gamma q(A'x+\gamma x)$ by (5.4), implying that $\gamma = 0$. Hence $n = 0$ and so (3) $A'^2x = 0$ follows as desired. □

Lemma 5.6. *If $B'A'^2y \neq 0$, then (1) $\nu = 0$, (2) $B'A'^2y = \mu(q+1)(q-1)A'^2y$, (3) $A'^3y = 0$, (4) $A'^2y \in \text{span}\{A'y, B'y, y\}$, (5) $B'^2y \in \text{span}\{A'y, B'y, y\}$, (6) $A'B'y \neq 0$.*

Proof. Since $B'(A'y) = 0$ but $B'A'(A'y) \neq 0$, (1), (2) and (3) follow from Lemma 5.5.

Now we show that $A'^2y \in \text{span}\{A'y, B'y, y\}$. Assume on the contrary that $A'^2y, A'y, B'y, y$ are $\mathbb{C}$-independent. Let $T \in \mathcal{B}_1$ such that $TA'^2y = Ty = A'y + y$, $TA'y = -(A'y+y)$ and $TB'y = 0$. Then $TB'A'(A'y+y) = \alpha(A'y+y)$, where $\alpha = \mu(q+1)(q-1) \neq 0$, and $TA'(A'y+y) = TB'(A'y+y) = T(A'y+y) = 0$, and so $H_1(T)(A'y+y) = \alpha(A'y+y)$ by (5.4), a contradiction. Therefore (4) $A'^2y \in \text{span}\{A'y, B'y, y\}$ follows.

Thus we may write $A'^2y = \xi A'y + \eta B'y + \zeta y$ for some $\xi, \eta, \zeta \in \mathbb{C}$. Since $A'^3y = 0$ but $A'^2y \neq 0$, then $A'^2y, A'y$ and y are $\mathbb{C}$-independent, and so $\eta \neq 0$. Comparing $B'A'^2y = \alpha A'^2y$ with $B'A'^2y = B'(\xi A'y + \eta B'y + \zeta y) = \eta B'^2y + \zeta B'y$, we obtain that $B'^2y \in \text{span}\{A'^2y, B'y\} \subseteq \text{span}\{A'y, B'y, y\}$ and so (5) holds.

From $0 = A'^3y = A'(\xi A'y + \eta B'y + \zeta y)$, it follows that $A'B'y = -\eta^{-1}(\xi A'^2y + \zeta A'y)$. Assume on the contrary $A'B'y = 0$. Then $\xi = \zeta = 0$, that is, $A'^2y = \eta B'y$, and so $B'A'^2y = \eta B'^2y$. On the other hand, $B'A'^2y = \alpha A'^2y = \alpha\eta B'y$, so $B'^2y = \alpha B'y$. Let $T \in \mathcal{B}_1$ such that $TA'y = y$ and $TB'y = Ty = 0$. Then $TA'(y+\alpha^{-1}q^2B'y) = y$ and $TB'A'(y+\alpha^{-1}q^2B'y) = TB'(y+\alpha^{-1}q^2B'y) = T(y+\alpha^{-1}q^2B'y) = 0$. Thus $H_1(T)(y+\alpha^{-1}q^2B'y) = -\alpha(y+\alpha^{-1}q^2B'y)$ by (5.4), a contradiction. Therefore (6) $A'B'y \neq 0$ follows. □

Since $B'A'x = BAx - \mu Ax - \nu Bx + \mu\nu x$, it follows from Lemma 5.3 that $B'A'x \in \text{span}\{Ax, Bx, x\} = \text{span}\{A'x, B'x, x\}$ for all $x \in X$. Moreover, we have

Lemma 5.7. *$B'A'x \in \text{span}\{A'y, B'y, y\}$ for all $x \in X$.*

Proof. Assume on the contrary that $B'A'x, A'y, B'y, y$ are $\mathbb{C}$-independent for some $x \in X$, then by [5, Lemma 2.1] there exists a nonzero $\lambda \in \mathbb{C}$ such that $B'A'x, A'y + \lambda A'x, B'y + \lambda B'x$ and $y + \lambda x$ are $\mathbb{C}$-independent. As $B'A'(y+\lambda x) = \lambda B'A'x$, it follows that $B'A'(y+\lambda x), A'(y+\lambda x), B'(y+\lambda x)$ and $y + \lambda x$ are $\mathbb{C}$-independent, contradicting Lemma 5.3. Hence, $B'A'x \in \text{span}\{A'y, B'y, y\}$ for every $x \in X$. □

Lemma 5.8. *Either $B'A' = 0$ or $\dim X \leqslant 3$.*

Proof. As a vector space cannot be the set-theoretic union of two proper subspaces, it suffices to show that, for each $x \in X$, either $B'A'x = 0$ or $x \in \text{span}\{A'y, B'y, y\}$. Let $x \in X$ such that $B'A'x \neq 0$. By

Lemma 5.3, for every $\xi \in \mathbb{C}$, there exist $\mu_\xi, \nu_\xi \in \mathbb{C}$ such that $(B-\mu_\xi I)(A-\nu_\xi I)(x+\xi y)=0$. From $(B-\mu_\xi I)(A-\nu_\xi I) = (B'-(\mu_\xi-\mu)I)(A'-(\nu_\xi-\nu)I)$, it follows that

$$\begin{aligned} B'A'x &= (\mu_\xi-\mu)A'x + (\nu_\xi-\nu)B'x - (\mu_\xi-\mu)(\nu_\xi-\nu)x \\ &\quad + \xi\big((\mu_\xi-\mu)A'y + (\nu_\xi-\nu)B'y - (\mu_\xi-\mu)(\nu_\xi-\nu)y\big). \end{aligned} \tag{5.5}$$

We prove first some properties about the coefficients μ_ξ, ν_ξ corresponding to ξ.

(1) *If $\mu_{\xi_1}=\mu_{\xi_2}$ and $\nu_{\xi_1}=\nu_{\xi_2}$ for $\xi_1,\xi_2\in\mathbb{C}$, then $\xi_1=\xi_2$.*

Assume on the contrary $\xi_1\neq\xi_2$. The difference of the expressions with ξ_1 and ξ_2, respectively, in place of ξ in (5.5) yields

$$(\xi_1-\xi_2)\big((\mu_{\xi_1}-\mu)A'y + (\nu_{\xi_1}-\nu)B'y - (\mu_{\xi_1}-\mu)(\nu_{\xi_1}-\nu)y\big)=0.$$

Then the $\mathbb{C}$-independence of $A'y$, $B'y$ and y implies $\mu_{\xi_1}-\mu=\nu_{\xi_1}-\nu=0$, and so $B'A'x=0$ by (5.5), a contradiction. Hence, $\xi_1=\xi_2$.

(2) *If $\nu_{\xi_1}=\nu_{\xi_2}=\nu$ or $\mu_{\xi_1}=\mu_{\xi_2}=\mu$ for $\xi_1,\xi_2\in\mathbb{C}$, then $\xi_1=\xi_2$.*

Assume on the contrary $\xi_1\neq\xi_2$. Suppose first $\nu_{\xi_1}=\nu_{\xi_2}=\nu$; then $\mu_{\xi_1}\neq\mu_{\xi_2}$ by (1). It follows from (5.5) that

$$B'A'x = (\mu_{\xi_1}-\mu)(A'x+\xi_1A'y) = (\mu_{\xi_2}-\mu)(A'x+\xi_2A'y),$$

which implies $A'x\in\mathbb{C}A'y$. From $B'A'y=0$, it follows that $B'A'x=0$, a contradiction.

Suppose next $\mu_{\xi_1}=\mu_{\xi_2}=\mu$; then $\nu_{\xi_1}\neq\nu_{\xi_2}$ by (1). It follows from (5.5) that

$$B'A'x = (\nu_{\xi_1}-\nu)(B'x+\xi_1B'y) = (\nu_{\xi_2}-\nu)(B'x+\xi_2B'y),$$

which implies that $B'x=\omega B'y$ for some $\omega\in\mathbb{C}$ and so $B'A'x=\tau B'y$ for some nonzero $\tau\in\mathbb{C}$. Now $B'(x-\omega y)=0$ but $B'A'(x-\omega y)=B'A'x\neq 0$, so we have by Lemma 5.5 that (i) $\nu=0$, (ii) $B'A'^2y=\alpha A'^2y$, (iii), $B'A'(x-\omega y)=\alpha A'(x-\omega y)$ and (iv) $A'^2(x-\omega y)=0$, where $\alpha=\mu(q+1)(q-1)\neq 0$. From (iii) and $B'A'x=\tau B'y$, it follows that $\tau B'y=\alpha A'(x-\omega y)$ and so $A'B'y=0$ by (iv). Then $B'A'^2y=0$ by Lemma 5.6, and so $A'^2y=0$ by (ii). Let $T\in\mathcal{B}_1$ such that $TA'y=y$; then $H_1(T)A'y=\alpha qA'y$ by (5.3), a contradiction. Therefore, $\xi_1=\xi_2$ in either situation.

(3) *If $\mu_{\xi_0}=\mu_{\xi_1}=\mu_{\xi_2}$ or $\nu_{\xi_0}=\nu_{\xi_1}=\nu_{\xi_2}$ for $\xi_0,\xi_1,\xi_2\in\mathbb{C}$, then ξ_0,ξ_1,ξ_2 are not all distinct.*

Consider first the situation that $\mu_{\xi_0}=\mu_{\xi_1}=\mu_{\xi_2}$. The difference of the expressions with ξ_i and ξ_j, respectively, in place of ξ in (5.5) yields

$$\begin{aligned} &(\nu_{\xi_i}-\nu_{\xi_j})(B'x-(\mu_{\xi_0}-\mu)x) = (\xi_j-\xi_i)(\mu_{\xi_0}-\mu)A'y \\ &+ \big(\xi_j(\nu_{\xi_j}-\nu)-\xi_i(\nu_{\xi_i}-\nu)\big)B'y - (\mu_{\xi_0}-\mu)\big(\xi_j(\nu_{\xi_j}-\nu)-\xi_i(\nu_{\xi_i}-\nu)\big)y \end{aligned}$$

for all $i,j=0,1,2$. If ξ_0,ξ_1,ξ_2 are all distinct, then so are $\nu_{\xi_0},\nu_{\xi_1},\nu_{\xi_2}$ by (1). Hence,

$$\begin{aligned} B'x-(\mu_{\xi_0}-\mu)x = (\nu_{\xi_i}-\nu_{\xi_0})^{-1}\Big(&(\xi_0-\xi_i)(\mu_{\xi_0}-\mu)A'y + \big(\xi_0(\nu_{\xi_0}-\nu) \\ &-\xi_i(\nu_{\xi_i}-\nu)\big)B'y - (\mu_{\xi_0}-\mu)\big(\xi_0(\nu_{\xi_0}-\nu)-\xi_i(\nu_{\xi_i}-\nu)\big)y\Big) \end{aligned} \tag{5.6}$$

for $i=1,2$. Comparing the expressions with $i=1$ and $i=2$, respectively, in (5.6), we obtain

$$(\nu_{\xi_1}-\nu_{\xi_0})^{-1}(\xi_0-\xi_1)(\mu_{\xi_0}-\mu) = (\nu_{\xi_2}-\nu_{\xi_0})^{-1}(\xi_0-\xi_2)(\mu_{\xi_0}-\mu), \tag{5.7}$$

and

$$(\nu_{\xi_1}-\nu_{\xi_0})^{-1}(\xi_0\nu_{\xi_0}-\xi_1\nu_{\xi_1}-\nu(\xi_0-\xi_1))=(\nu_{\xi_2}-\nu_{\xi_0})^{-1}(\xi_0\nu_{\xi_0}-\xi_2\nu_{\xi_2}-\nu(\xi_0-\xi_2)). \quad (5.8)$$

By (2) we have $\mu_{\xi_0}-\mu\neq 0$ and so (5.7) is reduced to

$$(\nu_{\xi_1}-\nu_{\xi_0})^{-1}(\xi_0-\xi_1)=(\nu_{\xi_2}-\nu_{\xi_0})^{-1}(\xi_0-\xi_2). \quad (5.9)$$

From (5.8) and (5.9) it follows that

$$(\nu_{\xi_1}-\nu_{\xi_0})^{-1}(\xi_0\nu_{\xi_0}-\xi_1\nu_{\xi_1})=(\nu_{\xi_2}-\nu_{\xi_0})^{-1}(\xi_0\nu_{\xi_0}-\xi_2\nu_{\xi_2}),$$

or equivalently

$$(\xi_0(\nu_{\xi_0}-\nu_{\xi_1})+\nu_{\xi_1}(\xi_0-\xi_1))(\nu_{\xi_2}-\nu_{\xi_0})=(\xi_0(\nu_{\xi_0}-\nu_{\xi_2})+\nu_{\xi_2}(\xi_0-\xi_2))(\nu_{\xi_1}-\nu_{\xi_0}),$$

and so

$$\nu_{\xi_1}(\xi_0-\xi_1)(\nu_{\xi_2}-\nu_{\xi_0})=\nu_{\xi_2}(\xi_0-\xi_2)(\nu_{\xi_1}-\nu_{\xi_0}).$$

From the last identity and (5.9), it follows that $(\xi_0-\xi_1)(\nu_{\xi_1}-\nu_{\xi_2})(\nu_{\xi_0}-\nu_{\xi_2})=0$, a contradiction. Therefore, ξ_0,ξ_1,ξ_2 are not all distinct. The proof for the situation $\nu_{\xi_0}=\nu_{\xi_1}=\nu_{\xi_2}$ is similar.

By (1)–(3), we may choose $\xi_1,\xi_2\in\mathbb{C}$ with $\xi_1\neq\xi_2$ such that $\nu_{\xi_1},\nu_{\xi_2},\nu$ are all distinct and also that $\mu_{\xi_1},\mu_{\xi_2},\mu$ are all distinct. Since $B'A'x\in\operatorname{span}\{A'y,B'y,y\}$ by Lemma 5.7, it follows from (5.5) that

$$(\mu_{\xi_1}-\mu)A'x+(\nu_{\xi_1}-\nu)B'x-(\mu_{\xi_1}-\mu)(\nu_{\xi_1}-\nu)x\in\operatorname{span}\{A'y,B'y,y\}$$

and

$$(\mu_{\xi_2}-\mu)A'x+(\nu_{\xi_2}-\nu)B'x-(\mu_{\xi_2}-\mu)(\nu_{\xi_2}-\nu)x\in\operatorname{span}\{A'y,B'y,y\}.$$

Eliminating $A'x$ and $B'x$, respectively, from the above two relations, we get

$$\Delta_{\xi_1,\xi_2}B'x-(\mu_{\xi_1}-\mu)(\mu_{\xi_2}-\mu)(\nu_{\xi_2}-\nu_{\xi_1})x\in\operatorname{span}\{A'y,B'y,y\} \quad (5.10)$$

and

$$\Delta_{\xi_1,\xi_2}A'x-(\nu_{\xi_1}-\nu)(\nu_{\xi_2}-\nu)(\mu_{\xi_1}-\mu_{\xi_2})x\in\operatorname{span}\{A'y,B'y,y\}, \quad (5.11)$$

where $\Delta_{\xi_1,\xi_2}=(\mu_{\xi_1}-\mu)(\nu_{\xi_2}-\nu)-(\mu_{\xi_2}-\mu)(\nu_{\xi_1}-\nu)$. Clearly, if $\Delta_{\xi_1,\xi_2}=0$, then $x\in\operatorname{span}\{A'y,B'y,y\}$ as desired. If $\Delta_{\xi_1,\xi_2}\neq 0$, then $A'x-\alpha_0x, B'x-\beta_0x\in\operatorname{span}\{A'y,B'y,y\}$ for some nonzero $\alpha_0,\beta_0\in\mathbb{C}$. From $A'x-\alpha_0x\in\operatorname{span}\{A'y,B'y,y\}$, it follows that $B'A'x-\alpha_0B'x\in\operatorname{span}\{B'^2y,B'y\}$ and so $B'x\in\operatorname{span}\{B'^2y,A'y,B'y,y\}$. Hence $x\in\operatorname{span}\{B'^2y,A'y,B'y,y\}$ follows from $B'x-\beta_0x\in\operatorname{span}\{A'y,B'y,y\}$. If $B'A'^2y\neq 0$, then $B'^2y\in\operatorname{span}\{A'y,B'y,y\}$ by Lemma 5.6 and consequently $x\in\operatorname{span}\{A'y,B'y,y\}$ as desired.

So we assume $B'A'^2y=0$. Set $x'=B'y$. Suppose first $B'A'x'\neq 0$. Proceed as before with x,μ_ξ,ν_ξ replaced by x',μ'_ξ,ν'_ξ. We claim that there exist ξ_1,ξ_2 such that $\mu'_{\xi_1},\mu'_{\xi_2},\mu$ are all distinct, $\nu'_{\xi_1},\nu'_{\xi_2},\nu$ are all distinct and $\Delta'_{\xi_1,\xi_2}=(\mu'_{\xi_1}-\mu)(\nu'_{\xi_2}-\nu)-(\mu'_{\xi_2}-\mu)(\nu'_{\xi_1}-\nu)\neq 0$. Assume on the contrary $\Delta'_{\xi_i,\xi_j}=0$ for all such pair ξ_i,ξ_j. From the $'$-version of (5.5) we obtain

$$\begin{aligned}(\nu'_{\xi_j}-\nu)B'A'x' &= (\nu'_{\xi_j}-\nu)\Big((\mu'_{\xi_i}-\mu)A'x'+(\nu'_{\xi_i}-\nu)B'x'-(\mu'_{\xi_i}-\mu)(\nu'_{\xi_i}-\nu)x'\\ &\quad +\xi_i((\mu'_{\xi_i}-\mu)A'y+(\nu'_{\xi_i}-\nu)B'y-(\mu'_{\xi_i}-\mu)(\nu'_{\xi_i}-\nu)y)\Big)\end{aligned}$$

and

$$(\nu'_{\xi_i}-\nu)B'A'x' = (\nu'_{\xi_i}-\nu)\Big((\mu'_{\xi_j}-\mu)A'x' + (\nu'_{\xi_j}-\nu)B'x' - (\mu'_{\xi_j}-\mu)(\nu'_{\xi_j}-\nu)x' + \xi_j((\mu'_{\xi_j}-\mu)A'y + (\nu'_{\xi_j}-\nu)B'y - (\mu'_{\xi_j}-\mu)(\nu'_{\xi_j}-\nu)y)\Big).$$

The difference of them gives

$$\begin{aligned}(\nu'_{\xi_j}-\nu'_{\xi_i})B'A'(B'y) &= (\xi_i(\mu'_{\xi_i}-\mu)(\nu'_{\xi_j}-\nu) - \xi_j(\mu'_{\xi_j}-\mu)(\nu'_{\xi_i}-\nu))A'y \\ &\quad + (\nu'_{\xi_i}-\nu)(\nu'_{\xi_j}-\nu)((\xi_i-\xi_j) - (\mu'_{\xi_i}-\mu'_{\xi_j}))B'y \\ &\quad + (\nu'_{\xi_i}-\nu)(\nu'_{\xi_j}-\nu)(\mu(\xi_i-\xi_j) + (\xi_j\mu'_{\xi_j}-\xi_i\mu'_{\xi_i}))y.\end{aligned} \tag{5.12}$$

Choose $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$ such that $\nu'_{\xi_1}, \nu'_{\xi_2}, \nu'_{\xi_3}, \nu$ are all distinct and $\mu'_{\xi_1}, \mu'_{\xi_2}, \mu'_{\xi_3}, \mu$ are all distinct too. Set $i = 1$ and $j = 2, 3$, respectively, in (5.12). Comparing the corresponding coefficients of the resulted identities, we have

$$\begin{aligned}&(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\xi_1(\mu'_{\xi_1}-\mu)(\nu'_{\xi_2}-\nu) - \xi_2(\mu'_{\xi_2}-\mu)(\nu'_{\xi_1}-\nu)) \\ &\quad = (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\xi_1(\mu'_{\xi_1}-\mu)(\nu'_{\xi_3}-\nu) - \xi_3(\mu'_{\xi_3}-\mu)(\nu'_{\xi_1}-\nu)),\end{aligned} \tag{5.13}$$

$$\begin{aligned}&(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_2}-\nu)((\xi_1-\xi_2) - (\mu'_{\xi_1}-\mu'_{\xi_2})) \\ &\quad = (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_3}-\nu)((\xi_1-\xi_3) - (\mu'_{\xi_1}-\mu'_{\xi_3})),\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}&(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_2}-\nu)(\mu(\xi_1-\xi_2) + (\xi_2\mu'_{\xi_2}-\xi_1\mu'_{\xi_1})) \\ &= (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_3}-\nu)(\mu(\xi_1-\xi_3) + (\xi_3\mu'_{\xi_3}-\xi_1\mu'_{\xi_1})).\end{aligned} \tag{5.15}$$

Since $(\mu'_{\xi_1}-\mu)(\nu'_{\xi_j}-\nu) = (\mu'_{\xi_j}-\mu)(\nu'_{\xi_1}-\nu)$, (5.13) is reduced to

$$(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_2}-\nu)(\xi_1-\xi_2) = (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_3}-\nu)(\xi_1-\xi_3). \tag{5.16}$$

By (5.16) it follows from (5.14) that

$$(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_2}-\nu)(\mu'_{\xi_1}-\mu'_{\xi_2}) = (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_3}-\nu)(\mu'_{\xi_1}-\mu'_{\xi_3}). \tag{5.17}$$

Comparing (5.16) and (5.17), we get

$$(\xi_1-\xi_3)(\mu'_{\xi_1}-\mu'_{\xi_2}) = (\xi_1-\xi_2)(\mu'_{\xi_1}-\mu'_{\xi_3}),$$

or equivalently

$$\xi_3(\mu'_{\xi_1}-\mu'_{\xi_2}) + \xi_1(\mu'_{\xi_2}-\mu'_{\xi_3}) + \xi_2(\mu'_{\xi_3}-\mu'_{\xi_1}) = 0. \tag{5.18}$$

Also, by (5.16) it follows from (5.15) that

$$(\nu'_{\xi_2}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_2}-\nu)(\xi_2\mu'_{\xi_2}-\xi_1\mu'_{\xi_1}) = (\nu'_{\xi_3}-\nu'_{\xi_1})^{-1}(\nu'_{\xi_3}-\nu)(\xi_3\mu'_{\xi_3}-\xi_1\mu'_{\xi_1}).$$

Comparing this with (5.16), we get

$$(\xi_1-\xi_3)(\xi_2\mu'_{\xi_2}-\xi_1\mu'_{\xi_1})=(\xi_1-\xi_2)(\xi_3\mu'_{\xi_3}-\xi_1\mu'_{\xi_1}),$$

or equivalently

$$\xi_1\xi_2(\mu'_{\xi_1}-\mu'_{\xi_2})+\xi_2\xi_3(\mu'_{\xi_2}-\mu'_{\xi_3})+\xi_3\xi_1(\mu'_{\xi_3}-\mu'_{\xi_1})=0. \tag{5.19}$$

From the trivial identity

$$(\mu'_{\xi_1}-\mu'_{\xi_2})+(\mu'_{\xi_2}-\mu'_{\xi_3})+(\mu'_{\xi_3}-\mu'_{\xi_1})=0$$

together with (5.18) and (5.19) it follows that $(\xi_1-\xi_2)(\xi_2-\xi_3)(\xi_3-\xi_1)=0$, contradicting our choice of ξ_1,ξ_2,ξ_3. Therefore, there exist ξ_1,ξ_2 such that $\mu'_{\xi_1},\mu'_{\xi_2},\mu$ are all distinct, $\nu'_{\xi_1},\nu'_{\xi_2},\nu$ are all distinct and $\Delta'_{\xi_1,\xi_2}\neq 0$. Then $B'^2y=B'x'\in\operatorname{span}\{A'y,B'y,y\}$ by the $'$-version of (5.10), and so $x\in\operatorname{span}\{B'^2y,A'y,B'y,y\}\subseteq\operatorname{span}\{A'y,B'y,y\}$ as desired.

Suppose next that $B'A'x'=B'A'B'y=0$. From $x\in\operatorname{span}\{B'^2y,A'y,B'y,y\}$, it follows that $B'A'x\in\mathbb{C}B'A'B'^2y$ since $B'A'^2y=0$. Thus $B'A'B'^2y\neq 0$. Set $x''=B'^2y$; then $B'A'x''\neq 0$. Proceed as before with x,μ_ξ,ν_ξ replaced by x'',μ''_ξ,ν''_ξ. If there exist ξ_1,ξ_2 such that $\mu''_{\xi_1},\mu''_{\xi_2},\mu$ are all distinct, $\nu''_{\xi_1},\nu''_{\xi_2},\nu$ are all distinct and $\Delta''_{\xi_1,\xi_2}=(\mu''_{\xi_1}-\mu)(\nu''_{\xi_2}-\nu)-(\mu''_{\xi_2}-\mu)(\nu''_{\xi_1}-\nu)=0$, then $B'^2y=x''\in\operatorname{span}\{A'y,B'y,y\}$ by the $''$-version of (5.10), and so $x\in\operatorname{span}\{A'y,B'y,y\}$ as desired.

The proof will be complete if we show that the assumption $\Delta''_{\xi_1,\xi_2}\neq 0$ for all such pair ξ_1,ξ_2 will lead to a contradiction. In this case, $A'B'^2y-\alpha_0B'^2y\in\operatorname{span}\{A'y,B'y,y\}$ for some nonzero $\alpha_0\in\mathbb{C}$ by the $''$-version of (5.11), and so $B'A'^2B'^2y=\alpha_0B'A'B'^2y\neq 0$. Let $x'''=A'B'^2y$; then $B'A'x'''\neq 0$. Proceed as before with x,μ_ξ,ν_ξ replaced by $x''',\mu'''_\xi,\nu'''_\xi$. If there exist ξ_1,ξ_2 such that $\mu'''_{\xi_1},\mu'''_{\xi_2},\mu$ are all distinct, $\nu'''_{\xi_1},\nu'''_{\xi_2},\nu$ are all distinct and $\Delta'''_{\xi_1,\xi_2}=(\mu'''_{\xi_1}-\mu)(\nu'''_{\xi_2}-\nu)-(\mu'''_{\xi_2}-\mu)(\nu'''_{\xi_1}-\nu)=0$, then $x'''\in\operatorname{span}\{A'y,B'y,y\}$ by the $'''$-version of (5.10), and so $B'A'x'''=0$, a contradiction. Hence $\Delta'''_{\xi_1,\xi_2}\neq 0$ for all such pair ξ_1,ξ_2. So by the $'''$-version of (5.10) we have $B'A'B'^2y-\beta_0A'B'^2y\in\operatorname{span}\{A'y,B'y,y\}$ for some nonzero $\beta_0\in\mathbb{C}$. Since $B'A'B'^2y\in\operatorname{span}\{A'y,B'y,y\}$ by Lemma 5.7, it follows that $A'B'^2y\in\operatorname{span}\{A'y,B'y,y\}$, and so $B'A'^2B'^2y=0$, a contradiction again. $\square$

6. Proof of Theorem 5.2

As we have seen before, if $\dim X\leq 3$ then $L(X)\simeq M_3(\mathbb{C})$ and so $H(T)^3=0$ follows from the quasi-nilpotency of $H(T)$ for all $T\in\mathcal{B}$. So we assume $\dim X\geqslant 4$ and hence $B'A'=0$ by Lemma 5.8. In this case (5.2) can be rewritten as

$$\begin{aligned}H_1(T)=&-(q^2B'+\alpha I)TA'-(qA'+\nu(q-1)I)TB'\\&+(q^3A'B'+\alpha qA'+\nu q^2(q-1)B'+\alpha\nu(q-1)I)T\end{aligned} \tag{6.1}$$

for $T\in\mathcal{B}_1$, where $\alpha=\mu(q+1)(q-1)$. We claim that it suffices to consider the case $\alpha=\nu(q-1)=0$. We proceed according as whether A'^2 and A' are locally dependent.

Case 1. A'^2x and $A'x$ are $\mathbb{C}$-independent for some $x\in X$.

Let $T\in\mathcal{B}_1$ such that $TA'^2x=A'x$ and $TA'x=0$. Then $H_1(T)A'x=-\alpha A'x$ by (6.1), implying $\alpha=0$ as desired. So (6.1) is reduced to

$$H_1(T)=-q^2B'TA'-(qA'+\nu(q-1)I)TB'+(q^3A'B'+\nu q^2(q-1)B')T \tag{6.2}$$

for $T \in \mathcal{B}_1$. Let $C = qA' + \nu(q-1)I$. Then $A'C = CA' = qA'^2 + \nu(q-1)A'$, $B'C = \nu(q-1)B'$ and $CB' = qA'B' + \nu(q-1)B'$. Then (6.2) can be simplified as

$$H_1(T) = -q^2B'TA' - CTB' + q^2CB'T \tag{6.3}$$

for $T \in \mathcal{B}_1$.

Now we proceed to show either $\nu(q-1) = 0$ or $H_1(T)^3 = 0$ for $T \in \mathcal{B}_1$. Assume on the contrary $\nu(q-1) \neq 0$. Consider first the situation $A'B' = 0$. Then $CB' = B'C = \nu(q-1)B'$ and so

$$H_1(T) = -CTB' + q^2B'T(\nu(q-1)I - A').$$

If $B'Cz \notin \mathbb{C}(\nu(q-1)I - A')Cz$ for some $z \in X$, there would exist $T \in \mathcal{B}_1$ such that $TB'Cz = z$ and $T(\nu(q-1)I - A')Cz = 0$, then $H_1(T)Cz = -Cz$, a contradiction. Hence $B'Cx \in \mathbb{C}(\nu(q-1)I - A')Cx$ for all $x \in X$. By [7, Lemma 2.7], $B'C = \beta(\nu(q-1)I - A')C$ for some nonzero $\beta \in \mathbb{C}$. Thus $\nu(q-1)B'^2 = B'^2C = \beta B'(\nu(q-1)I - A')C = \beta\nu(q-1)B'C$ and so $B'^2 = \beta B'C$. Let $x \in X$ such that $B'x \neq 0$ and let $T \in \mathcal{B}_1$ such that $TB'x = B'x$. Then $H_1(T)B'x = \beta(\nu(q-1))^2(q^2-1)B'x$. This means $q+1 = 0$ as $\nu(q-1) \neq 0$. Thus $C = qA' + \nu(q-1)I = \nu(q-1)I - A'$. From $B'C = \beta(\nu(q-1)I - A')C$ it follows that $B'C = \beta C^2$. Using $B'C = CB' = \beta C^2 = \beta^{-1}B'^2 = \nu(q-1)B'$, $q = -1$ and $H_1(T) = -CTB' + B'TC$, we readily verify $H_1(T)^3 = 0$ for all $T \in \mathcal{B}_1$ and so we are done.

Consider next the situation $A'B' \neq 0$. If $A'B'x \notin B'X$ for some $x \in X$, there would exist $T \in \mathcal{B}_1$ such that $TA'B'x = x$ and $TB'x = TB'^2x = 0$, and so $H_1(T)B'x = -q^2B'x$ by (6.3), a contradiction. Thus $A'B'X \subseteq B'X$ and then $CB'X \subseteq B'X$. In view of (6.3) we have $H_1(T)X \subseteq B'X + CTB'X$. Thus if $\mathrm{rank}(B') \leq 1$, then $\mathrm{rank}H_1(T) \leq 2$ for each $T \in \mathcal{B}_1$. Then $H_1(T)$ is nilpotent, whence $H_1(T)^3 = 0$ for all $T \in \mathcal{B}_1$ and so we are done. Hence we may assume $\mathrm{rank}(B') \geq 2$.

If $A'Cz$, $B'Cz$ and Cz are $\mathbb{C}$-independent for some $z \in X$, then there would exist $T \in \mathcal{B}_1$ such that $TA'Cz = TCz = 0$ and $TB'Cz = z$, and so $H_1(T)Cz = -Cz$ by (6.3), a contradiction. Hence $A'Cz$, $B'Cz$ and Cz are $\mathbb{C}$-dependent for any $z \in X$. By [5, Theorem 2.4], there are three cases to be investigated.

Subcase 1. There exists a subspace V of X with $\dim V \leqslant 3$ such that $A'CX \cup B'CX \cup CX \subseteq V$.

Since $B'C = \nu(q-1)B' \neq 0$, so $B'X = B'CX \subseteq V$, $\mathrm{rank}(B') \leq \mathrm{rank}(C)$ and B' maps CX *onto* $B'CX = B'X$. The $\mathbb{C}$-independence of A'^2x and $A'x$ implies $A'Cx = qA'^2x + \nu(q-1)A'x \neq 0$. Since $B'A'Cx = 0$ and $A'Cx = CA'x \in CX$, the kernel of $B' : CX \to B'X$ is not 0. Hence $\mathrm{rank}(B') < \mathrm{rank}(C)$. Then $\mathrm{rank}(B') = 2$ and $\mathrm{rank}(C) = 3$, that is $CX = V$. Thus $\mathrm{kernel}(B') \cap V$ has dimension 1. Since $B'A'C = 0$, we have $\dim(A'CX) \leq 1$. Recall that $A'B'X \subseteq B'X$. From $0 \neq A'B'X \subseteq A'V = A'CX$, it follows that $A'CX = A'B'X \subseteq B'X$. Now $\mathrm{rank}(A'C) = 1$ and $\mathrm{rank}(B'C) = \mathrm{rank}(B) = 2$, so $A'C$ and $B'C$ are not locally $\mathbb{C}$-dependent by [5, Theorem 2.8]. That is, there exists $x_0 \in X$ such that $A'Cx_0$ and $B'Cx_0$ are $\mathbb{C}$-independent. Since $A'Cx_0$, $B'Cx_0$ and Cx_0 are $\mathbb{C}$-dependent, $Cx_0 \in \mathrm{span}\{A'Cx_0, B'Cx_0\} \subseteq A'CX + B'CX \subseteq B'X$. For any $z \in X$, there exists a nonzero $\lambda \in \mathbb{C}$ such that $A'C(x_0 + \lambda z)$ and $B'C(x_0 + \lambda z)$ are $\mathbb{C}$-independent by [5, Lemma 2.1]. As above, the $\mathbb{C}$-dependence of $A'C(x_0 + \lambda z)$, $B'C(x_0 + \lambda z)$ and $C(x_0 + \lambda z)$ yields $C(x_0 + \lambda z) \in B'X$ and so $Cz \in B'X$ for any $z \in X$. Thus $CX \subseteq B'X$, a contradiction.

Subcase 2. There exist $\theta_1, \theta_2, \theta_3 \in \mathbf{C}$, not all zero, such that $\theta_1A'C + \theta_2B'C + \theta_3C = 0$.

That is,

$$\theta_1(qA'^2 + \nu(q-1)A') + \theta_2\nu(q-1)B' + \theta_3(qA' + \nu(q-1)I) = 0. \tag{6.4}$$

If $\theta_2 = 0$, then, applying B' from the left to (6.4), we get $\theta_3\nu(q-1)B' = 0$. Hence $\theta_3 = 0$ and so $qA'^2 + \nu(q-1)A' = 0$, contradicting the fact that A'^2x and $A'x$ are $\mathbb{C}$-independent. Therefore $\theta_2 \neq 0$ and then $B' \in \mathrm{span}\{A'^2, A', I\}$ by (6.4). So $A'B' = B'A' = 0$, a contradiction.

Subcase 3. There exists an idempotent $E \in L(X)$ of rank one such that $\mathrm{span}\{(I-E)A'C, (I-E)B'C, (I-E)C\}$ has dimension 1.

Since $C = qA' + \nu(q-1)I = qA - \nu I$, we may write $A = \gamma I + A_1$, where $\gamma = \nu q^{-1}$ and $A_1 = q^{-1}C$. If $(I-E)C = 0$, then $A_1 = q^{-1}EC$ has rank at most 1. So $H(T) = P[P[T, B], A] = P[P[T, B], A_1]$ has rank at most 2. Thus $H(T)^3 = 0$ for all $T \in \mathcal{B}$ and we are done. Assume $(I-E)C \neq 0$; then $(I-E)B'C = \theta(I-E)C$ for some $\theta \in \mathbb{C}$. If $\theta = 0$, then $B' = (\nu(q-1))^{-1}B'C = (\nu(q-1))^{-1}EB'C$ has rank at most 1, contrary to our assumption. Hence, $\theta \neq 0$. Then $(I-E)A'C = (I-E)CA' = \theta^{-1}(I-E)B'CA' = \theta^{-1}(I-E)B'A'C = 0$. Thus $A'C = EA'C$ and so $A'CX \subseteq EX$. Since $0 \neq A'Cx \in A'CX$ and EX has dimension 1, we have $A'CX = EX$. From $A'^2C = A'CA' = EA'CA'$, it follows that $A'EX = A'^2CX \subseteq EX$. Now $B'C = \theta C + E(B'C - \theta C)$, so $A'B'X = A'B'CX \subseteq A'CX + A'EX \subseteq EX$. Hence $A'B'X = EX$ and $A'B'$ has rank 1. Then $B'E = 0$ since $B'EX = B'A'B'X = 0$, and so $B'^2C = B'(\theta C + E(B'C - \theta C)) = \theta B'C$. From $B'C = \nu(q-1)B'$ it follows that $B'^2 = (\nu(q-1))^{-1}B'^2C = (\nu(q-1))^{-1}\theta B'C = \theta B'$. If $A'B'z \notin \mathbb{C}B'z$ for some $z \in X$, there would exist $T \in \mathcal{B}_1$ such that $TA'B'z = z$ and $TB'z = 0$; then $TB'^2z = 0$ and $H_1(T)B'z = -q^2B'z$ by (6.3), a contradiction. Therefore $A'B'z \in \mathbb{C}B'z$ for all $z \in X$, so $A'B' = \lambda B'$ for some nonzero $\lambda \in \mathbb{C}$ by [7, Lemma 2.7]. Thus B' has rank 1, contrary to our assumption.

Case 2. A'^2z and $A'z$ are $\mathbb{C}$-dependent for all $z \in X$.

In this case, $A'^2 = \gamma A'$ for some $\gamma \in \mathbb{C}$ by [7, Lemma 2.7]. Assume first $\gamma \neq 0$. Since $A'y$, $B'y$ and y are $\mathbb{C}$-independent, so are $(A' - \gamma I)y$ and $B'(A' - \gamma I)y$. Let $T \in \mathcal{B}_1$ such that $TB'(A' - \gamma I)y = (A' - \gamma I)y$ and $T(A' - \gamma I)y = 0$. Then $H_1(T)(A' - \gamma I)y = -\nu(q-1)(A' - \gamma I)y$ by (6.1), implying $\nu(q-1) = 0$ as desired. Then (6.1) is reduced to

$$H_1(T) = -(q^2B' + \alpha I)TA' - qA'TB' + qA'(q^2B' + \alpha I)T \tag{6.5}$$

for $T \in \mathcal{B}_1$. Let $T \in \mathcal{B}_1$ such that $TA'y = A'y$. Then $H_1(T)A'y = \alpha\gamma(q-1)A'y$ by (6.5) and so $\alpha = 0$, as desired.

Assume next $\gamma = 0$, that is $A'^2 = 0$. Let $T \in \mathcal{B}_1$ such that $TA'y = A'y$. Then $H_1(T)A'y = \alpha\nu(q-1)A'y$ by (6.1) and so $\alpha\nu(q-1) = 0$. Hence $\alpha = 0$ or $\nu(q-1) = 0$.

Suppose first $\alpha = 0$; then (6.1) is reduced to (6.3) where $C = qA' + \nu(q-1)I$. Let $T \in \mathcal{B}_1$ such that $TA'y = Ty = 0$ and $TB'y = y$. Then $TA'Cy = TCy = 0$ and $TB'Cy = \nu(q-1)y$, and so by (6.3) $H_1(T)Cy = -\nu(q-1)Cy$, implying $\nu(q-1) = 0$ as desired.

Suppose next $\nu(q-1) = 0$; then (6.1) is reduced to (6.5). We claim that $A'(q^2B' + \alpha I) = 0$. Otherwise, let $x \in X$ with $A'(q^2B' + \alpha I)x \neq 0$ and $T \in \mathcal{B}_1$ such that $TA'(q^2B' + \alpha I)x = x$; then $H_1(T)A'(q^2B' + \alpha I)x = qA'(q^2B' + \alpha I)x$ by (6.5), a contradiction. Hence, $A'(q^2B' + \alpha I) = 0$ and so (6.5) is reduced to

$$H_1(T) = -(q^2B' + \alpha I)TA' - qA'TB'$$

for $T \in \mathcal{B}_1$. Now it is easy to see that $H_1(T)^3 = 0$ for all $T \in \mathcal{B}_1$ since $B'A' = A'^2 = 0$ and so we are done.

Now it remains to consider the case $\alpha = \nu(q-1) = 0$. In this case, (6.1) is reduced to

$$H_1(T) = -q^2B'TA' - qA'TB' + q^3A'B'T \tag{6.6}$$

for $T \in \mathcal{B}_1$. Suppose first $A'B' = 0$; then (6.6) is reduced to

$$H_1(T) = -qA'TB' - q^2B'TA' \tag{6.7}$$

for $T \in \mathcal{B}_1$. We claim that either $A'^2 = 0$ or $B'^2 = 0$, whence $H_1(T)^3 = 0$ for all $T \in \mathcal{B}_1$. Assume on the contrary that both $A'^2 \neq 0$ and $B'^2 \neq 0$; then there exists $x \in X$ such that $A'^2x \neq 0$ and $B'^2x \neq 0$. Since $B'A' = 0$, $A'x$ and $B'x$ are $\mathbb{C}$-independent. If A'^2x and B'^2x are also $\mathbb{C}$-independent, there would exist $T \in \mathcal{B}_1$ such that $TA'^2x = x$ and $TB'^2x = qx$, and then $H_1(T)(A'x + B'x) = -q^2(A'x + B'x)$ by

(6.7), a contradiction. Hence ${A'}^2x = \tau {B'}^2x$ for some nonzero $\tau \in \mathbb{C}$. Write $\tau = \rho^2$ and $q = p^2$ for some $\rho, p \in \mathbb{C}$. Take $T \in \mathcal{B}_1$ such that $T{B'}^2x = x$. Then

$$H_1(T)\left(B'x + \rho^{-1}p^{-1}A'x\right) = -\rho p^3\left(B'x + \rho^{-1}p^{-1}A'x\right),$$

by (6.7), a contradiction again. Therefore, either ${A'}^2 = 0$ or ${B'}^2 = 0$ and we are done.

Suppose next $A'B' \neq 0$. If ${A'}^2B' = 0$ or ${A'}^2B'x \notin \mathbb{C}A'B'x$ for some $x \in X$, there would exist $x \in X$ and $T \in \mathcal{B}_1$ such that $A'B'x \neq 0$, $T{A'}^2B'x = 0$ and $TA'B'x = x$, and then $H_1(T)A'B'x = q^3A'B'x$ by (6.6), a contradiction. Hence, ${A'}^2B' = \lambda A'B' \neq 0$ for some $\lambda \in \mathbb{C}$ by [7, Lemma 2.7]. If $A'B'x \notin B'X$ for some $x \in X$, there would exist $T \in \mathcal{B}_1$ such that $TA'B'x = x$ and $TB'x = T{B'}^2x = 0$, and then $H_1(T)B'x = -q^2B'x$ by (6.6), a contradiction. So $A'B'X \subseteq B'X$. Let $z \in X$ with $A'B'z \neq 0$. Then $A'B'z = B'z_0$ for some $z_0 \in X$ and then $A'B'z_0 = {A'}^2B'z = \lambda A'B'z = \lambda B'z_0$ and ${B'}^2z_0 = 0$. Let $T \in \mathcal{B}_1$ such that $TB'z_0 = z_0$. Then $H_1(T)B'z_0 = \lambda q^2(q-1)B'z_0$ by (6.6), implying that $q = 1$. So $H_1(T) = A'B'T - A'TB' - B'TA' = [A', [B', T]]$ for $T \in \mathcal{B}_1$. Then the conclusion of Theorem 5.2 follows from [7, Theorem 1.2].

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